$R_x = 5$

$R_y = OA \sin 0^\circ + OB \sin 30^\circ + OC \sin 60^\circ + OD \sin 90^\circ + OE \sin 120^\circ$

$= 0 + \sqrt{3} \sin 30^\circ + 5 \sin 60^\circ + \sqrt{3} \sin 90^\circ + 2 \sin 120^\circ$

$= \sqrt{3} \cdot \frac{1}{2} + 5 \cdot \frac{\sqrt{3}}{2} + \sqrt{3} \cdot 1 + 2 \cdot \frac{\sqrt{3}}{2}$

$R_y = 5\sqrt{3}$

$R = \sqrt{R_x^2 + R_y^2}$

$= \sqrt{25 + (25 \times 3)}$

$R = 10$

Also $\tan\theta = \frac{R_y}{R_x} = \sqrt{3}$

$\theta = 60^\circ$

Resultant acts towards the opposite angular point ie) along OC.

Equilibrium of a particle:

When the resultant of the forces acting at a point is zero, then the forces are said to be in equilibrium. In this case the particle is at rest inspite of the forces.

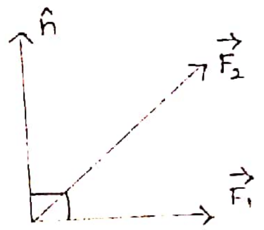
Equilibrium of a particle under three forces:

First we consider cases in which a particle is in equilibrium under the action of three forces.

Book Work : 1

To show that, if three forces keep a particle in equilibrium, then the forces are Coplanar.

Let the forces $\vec{F_1}, \vec{F_2}, \vec{F_3}$ keep a particle in equilibrium. Then the resultant force on the particle is $\vec{F_1} + \vec{F_2} + \vec{F_3}$.



Since the particle is in equilibrium,

$$\vec{F_1} + \vec{F_2} + \vec{F_3} = 0 \rightarrow (1)$$

Let $\hat{n}$ be the unit Vector perpendicular to both $\vec{F_1}$ & $\vec{F_2}$

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Then, $\hat{n} \cdot \vec{F_1} = 0$, $\hat{n} \cdot \vec{F_2} = 0$.

Now, multiplying (1) scalarly by $\hat{n}$, we see that

$$\hat{n} \cdot (\vec{F_1} + \vec{F_2} + \vec{F_3}) = 0$$

$$\hat{n} \cdot \vec{F_1} + \hat{n} \cdot \vec{F_2} + \hat{n} \cdot \vec{F_3} = 0$$

But $\hat{n} \cdot \vec{F_1} = 0$ and $\hat{n} \cdot \vec{F_2} = 0$. so $\hat{n} \cdot \vec{F_3} = 0$ which means that $\vec{F_3}$ also is perpendicular to $\hat{n}$. Therefore $\vec{F_1}, \vec{F_2}$ and $\vec{F_3}$ are Coplanar.

Book Work : 2 [Triangle of forces]

If three forces acting on a particle can be represented in magnitude & direction by the sides of a triangle, taken in order, then the forces keep the particle in equilibrium.

Let the given forces be represented in magnitude and direction by the sides AB, BC, CA of a triangle ABC.

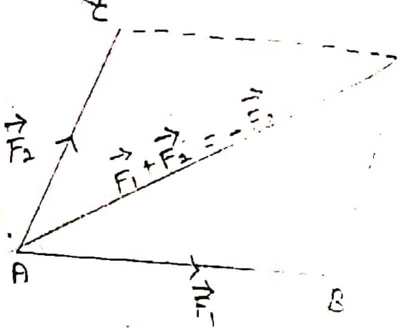
Then the forces are $\vec{AB}, \vec{BC}, \vec{CA}$

Their resultant is $\vec{AB} + \vec{BC} + \vec{CA}$. But, by Vector theory, $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$. Hence the resultant force acting on the particle is zero. So the particle is in equilibrium.

Book Work : 3 [Converse of triangle of forces]

If a particle is kept in equilibrium by three forces, then the forces can be represented in magnitude and direction by the sides of a triangle, taken in order.

Let the given forces be $\vec{F_1}, \vec{F_2}, \vec{F_3}$. They keep the particle in equilibrium. So their resultant is zero.



Hence,

$$\vec{F_1} + \vec{F_2} + \vec{F_3} = \vec{0}$$

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$$\vec{F_1} + \vec{F_2} = -\vec{F_3}$$

Let AB, AC denote $\vec{F_1}, \vec{F_2}$ in magnitude and direction. Complete the parallelogram ABPC. Then BP denotes $\vec{F_2}$ and AP denotes $\vec{F_1} + \vec{F_2}$ in magnitude and direction. But $\vec{F_1} + \vec{F_2} = -\vec{F_3}$. So AP denotes $-\vec{F_3}$. Hence PA denotes $\vec{F_3}$. This completes the proof.

Polygon of forces:

It can be easily seen that, if several coplanar forces acting on a particle can be represented in magnitude and direction by the sides of a polygon, taken in order, the forces keep the particle in equilibrium. This result is called the polygon of forces.

Converse of polygon of forces:

The Converse of polygon of forces is that, if a particle is kept in equilibrium by n forces, then they can be represented by the sides of a n-sided polygon. To prove the true of this, let us consider, in particular the six forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, \vec{F_4}, \vec{F_5}, \vec{F_6}$ which keeps a particle in equilibrium. Because of equilibrium,

$$\vec{F_1}+\vec{F_2}+\vec{F_3}+\vec{F_4}+\vec{F_5}+\vec{F_6}=\vec{0} \rightarrow (1)$$

Take points A, B, C, D, E, F such that

$$\vec{AB}=\vec{F_1}, \ \vec{BC}=\vec{F_2}, \ \vec{CD}=\vec{F_3}, \ \vec{DE}=\vec{F_4}, \ \vec{EF}=\vec{F_5}$$

Now, from vector theory, we have

$$\vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}+\vec{EF}+\vec{FA}=\vec{0} \rightarrow (2)$$

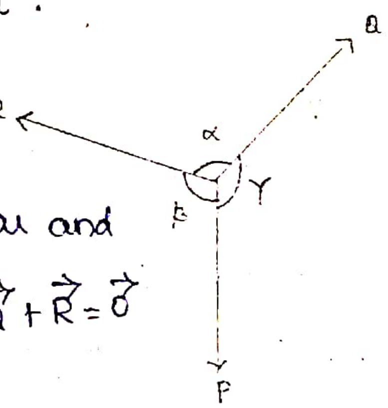
From (1) & (2) we get $\vec{FA}=\vec{F_6}$. This completes the proof.

Lami's theorem:

If a particle is in equilibrium under the action of three forces $\vec{P}, \vec{Q}, \vec{R}$ then to show that

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$$

where α is the angle between $\vec{P}$ and $\vec{R}$ and $\vec{Q}$ and $\vec{R}$, β is the angle between $\vec{P}$ and $\vec{Q}$. γ is the angle between



The forces keep the particle in equilibrium. So they are coplanar and their resultant is zero. Hence $\vec{P}+\vec{Q}+\vec{R}=\vec{0}$

Multiplying this Vectorially by $\vec{P}$,

i.e. $\vec{P}\times\vec{P}+\vec{P}\times\vec{Q}+\vec{P}\times\vec{R}=\vec{0} \rightarrow (1)$

Let $\hat{n}$ be the unit Vector perpendicular to the forces Such that $\vec{P},\vec{Q},\hat{n}$ form a right handed triad.

Then (1) becomes,

$$\vec{0}+PQ\sin\gamma\,\hat{n}+PR\sin\beta(-\hat{n})=\vec{0}$$

$$(PQ\sin\gamma-PR\sin\beta)\,\hat{n}=\vec{0}$$

$$PQ\sin\gamma-PR\sin\beta=0$$

$$Q\sin\gamma=R\sin\beta$$

$$\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma} \rightarrow (2)$$

Similarly the vectorial multiplication of $\vec{P}+\vec{Q}+\vec{R}=\vec{0}$ by $\vec{Q}$ gives

$$\frac{P}{\sin\alpha}=\frac{R}{\sin\gamma} \rightarrow (3)$$

From (2) & (3) we get

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}.$$

Equilibrium of a particle under several forces.

Now we study the situated in which a particle is in equilibrium under three or more forces

Book Work:

To prove that the necessary and sufficient Conditions for a system of Coplanar forces to keep a particle in equilibrium, is that the sums of the Components of the forces in two mutually perpendicular directions in the plane are zero.

Necessity part:

Let the forces be $\vec{F_1}, \vec{F_2}, \ldots \vec{F_n}$. Then the resultant force acting on the particle is $\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n}$

So, if the particle is in equilibrium, then

$$\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n}=0 \rightarrow (1)$$

Let $\vec{i}, \vec{j}$ be the unit vectors in two perpendicular directions. Multiplying (1) scalarly by $\vec{i}$ and $\vec{j}$

$$(\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n})\cdot\vec{i}=0, \quad (\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n})\cdot\vec{j}=0$$

$$\vec{F_1}\cdot\vec{i}+\vec{F_2}\cdot\vec{i}+\ldots+\vec{F_n}\cdot\vec{i}=0, \quad \vec{F_1}\cdot\vec{j}+\vec{F_2}\cdot\vec{j}+\ldots+\vec{F_n}\cdot\vec{j}=0$$

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This proves the necessity part that,

If the particle is in equilibrium, then the sums of the components of the forces in two mutually perpendicular directions are zero.

Sufficiency part

Now we have that the sums of the components are zeros. That is,

$$\vec{F_1}\cdot\vec{i}+\vec{F_2}\cdot\vec{i}+\ldots+\vec{F_n}\cdot\vec{i}=0$$

$$\vec{F_1}\cdot\vec{j}+\vec{F_2}\cdot\vec{j}+\ldots+\vec{F_n}\cdot\vec{j}=0$$

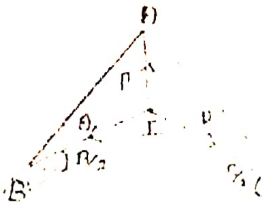
$$\Rightarrow (\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n})\cdot\vec{i}=0, \quad (\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n})\cdot\vec{j}=0$$

That is, the resultant force is either perpendicular to both $\vec{i}$ & $\vec{j}$ (or) is a zero force. The perpendicularity cannot happen. Thus $\vec{F_1}+\vec{F_2}+\ldots+\vec{F_n}=\vec{0}$.

Problems:

D1) I is the incentre of a triangle ABC. If forces of magnitudes P, Q, R acting along the bisectors IA, IB, IC are in equilibrium. S.T $\frac{P}{\cos A/2} = \frac{Q}{\cos B/2} =$

Soln:



The forces P, Q, R act at I and are in equilibrium. So we shall use Lami's theorem. The angles opposite to P, Q, R are $\angle BIC$, $\angle CIA$, $\angle AIB$

$$\therefore \frac{P}{\sin BIC} = \frac{Q}{\sin CIA} = \frac{R}{\sin AIB} \rightarrow (1)$$

Now, from ΔBIC

$$\angle BIC = 180^\circ - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$\sin BIC = \sin\left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= \sin(90^\circ - A/2)$$

$$\sin BIC = \cos A/2$$

Illy

$$\sin CIA = \cos B/2$$

$$\sin AIB = \cos C/2$$

Thus (1) becomes

$$\frac{P}{\cos A/2} = \frac{Q}{\cos B/2} = \frac{R}{\cos C/2}$$

(or) $P : Q : R = \cos A/2 : \cos B/2 : \cos C/2$.

D2) I is the incentre of a triangle ABC. If the forces $\vec{IA}, \vec{IB}, \vec{IC}$ acting at I are in equilibrium S.T ABC is an equilateral triangle.

Soln:

The forces are $\vec{IA}, \vec{IB}, \vec{IC}$ acting at I are ~~Se~~ ~~[illegible]~~ 74

respectively P, Q, R and represented by IA, IB, IC we get

$$\frac{AI}{\cos A/2} = \frac{BI}{\cos B/2} = \frac{CI}{\cos C/2} \rightarrow (1)$$

But, if r is the radius of the incentre, then

$$\frac{r}{AI} = \sin\frac{A}{2} \Rightarrow AI = \frac{r}{\sin\frac{A}{2}}$$

$$\frac{r}{BI} = \sin\frac{B}{2} \Rightarrow BI = \frac{r}{\sin\frac{B}{2}}$$

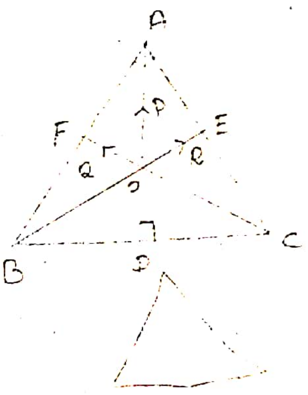
$$\frac{r}{CI} = \sin\frac{C}{2} \Rightarrow CI = \frac{r}{\sin\frac{C}{2}}$$

$$\therefore (1) \Rightarrow \frac{r}{\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{r}{\sin\frac{B}{2}\cos\frac{B}{2}} = \frac{r}{\sin\frac{C}{2}\cos\frac{C}{2}}$$

ie)
$$\frac{1}{\sin A} = \frac{1}{\sin B} = \frac{1}{\sin C}$$

which implies that $A = B = C$ and so the triangle is equileteral.

Q3) O is the orthocentre of a triangle ABC. If forces of magnitude P, Q, R acting along OA, OB, OC are in equilibrium, S.T $\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$.



Soln:

The forces P, Q, R act at O and are in equilibrium. So we shall use Lami's theorem. The angle opposite to P, Q, R are $\angle BOC$, $\angle COA$, $\angle AOB$

$$\frac{P}{\sin BOC} = \frac{Q}{\sin COA} = \frac{R}{\sin AOB} \rightarrow (1)$$

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If AD is the altitude through A

$\angle BOD = C$, $\angle COD = B$

$\angle BOC = B + C$

$\sin BOC = \sin(B+C) = \sin(180^\circ - A)$

$\sin BOC = \sin A$

III^{ly} $\sin COA = \sin B$

$\sin AOB = \sin C$

Thus ① becomes

$$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

But, by Sine formula, $\sin A : \sin B : \sin C = a : b : c$

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}.$$

D4) Three forces of magnitude P, Q, R acting at a point being parallel to the sides of a triangle are in equilibrium. If another set of forces of magnitudes P', Q', R' acting at a point being parallel to the sides of the same triangle, are also in equilibrium S.T $\frac{P}{P'} = \frac{Q}{Q'} = \frac{R}{R'}$.

Soln:

Let ABC be the triangle. Then the forces acting at the first point are $P\hat{BC}$, $Q\hat{CA}$, $R\hat{AB}$.

They are in equilibrium $P\hat{BC} + Q\hat{CA} + R\hat{AB} = 0$

III^{ly} for second case, $P'\hat{BC} + Q'\hat{CA} + R'\hat{AB} = 0$

$$\Rightarrow \frac{P}{R}\hat{BC} + \frac{Q}{R}\hat{CA} = -\hat{AB}$$

and $$\frac{P'}{R'}\hat{BC} + \frac{Q'}{R'}\hat{CA} = -\hat{AB}$$

Subtracting we get

$$\left(\frac{P}{R}-\frac{P'}{R'}\right)\hat{BC}+\left(\frac{Q}{R}-\frac{Q'}{R'}\right)\hat{CA}=0$$

W.K.T, if $\vec{a}$ & $\vec{b}$ are two non-parallel vectors and if $l\vec{a}+m\vec{b}=0$, then $l=0$ & $m=0$.

$$\therefore \frac{P}{R}-\frac{P'}{R'}=0 \quad \& \quad \frac{Q}{R}-\frac{Q'}{R'}=0$$

$$\frac{P}{R}=\frac{P'}{R'} \quad \& \quad \frac{Q}{R}=\frac{Q'}{R'}$$

$$\therefore \frac{P}{P'}=\frac{Q}{Q'}=\frac{R}{R'}$$

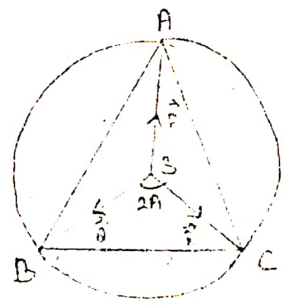
D5) S is the circumcentre of a triangle ABC. If forces of magnitudes P, Q, R acting along SA, SB, SC are in equilibrium. S.T P, Q, R are in the ratio

(i) $\dfrac{P}{\sin 2A}=\dfrac{Q}{\sin 2B}=\dfrac{R}{\sin 2C}$

(ii) $\dfrac{P}{a^2(b^2+c^2-a^2)}=\dfrac{Q}{b^2(c^2+a^2-b^2)}=\dfrac{R}{c^2(a^2+b^2-c^2)}$

(iii) $\dfrac{P}{\Delta BSC}=\dfrac{Q}{\Delta CSA}=\dfrac{R}{\Delta ASB}$

(iv) $P:Q:R=OA:OB:OC$ if O is the Centroid of the triangle.



Soln:

(i) S is the circumcentre of the ΔABC.

Forces P, Q, R act along SA, SB, SC and are in equilibrium.

We know that, $2\angle BAC=\angle BSC$

$\angle BSC = 2A$, $\angle CSA = 2B$, $\angle ASB = 2C$

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By Lami's theorem,

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

$$P : Q : R = \sin 2A : \sin 2B : \sin 2C$$

(ii) We know that,

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

$$\Rightarrow \frac{P}{2\sin A \cos A} = \frac{Q}{2\sin B \cos B} = \frac{R}{2\sin C \cos C} \rightarrow (1)$$

We have the cosine formula in ΔABC as

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2ac}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Also we have the Sine formula in ΔABC as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\sin A = a, \quad \sin B = b, \quad \sin C = c$$

$$(1) \Rightarrow \frac{P}{2a\left(\frac{b^2+c^2-a^2}{2bc}\right)} = \frac{Q}{2b\left(\frac{c^2+a^2-b^2}{2ac}\right)} = \frac{R}{2c\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

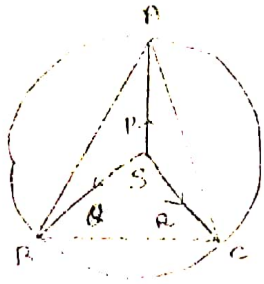
$\div$ by abc, we get

$$\frac{P}{a^2(b^2+c^2-a^2)} = \frac{Q}{b^2(c^2+a^2-b^2)} = \frac{R}{c^2(a^2+b^2-c^2)}$$

$$P : Q : R = a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$$

(iii) $\dfrac{P}{\sin 2A} = \dfrac{Q}{\sin 2B} = \dfrac{R}{\sin 2C} \rightarrow (2)$

$\angle BSC = 2A$



$\Delta BSC = \frac{1}{2} SB.SC \sin \angle BSC$

$\Delta BSC = \frac{1}{2} R^2 \sin 2A$

|||ly $\Delta CSA = \frac{1}{2} R^2 \sin 2B$

$\Delta ASB = \frac{1}{2} R^2 \sin 2C$

$\sin 2A = \dfrac{2\,\Delta BSC}{R^2}$, $\sin 2B = \dfrac{2\,\Delta CSA}{R^2}$, $\sin 2C = \dfrac{2\,\Delta ASB}{R^2}$

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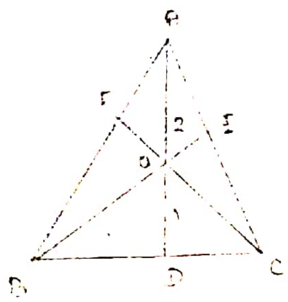
(2) $\Rightarrow \dfrac{P}{\Delta BSC} = \dfrac{Q}{\Delta CSA} = \dfrac{R}{\Delta ASB}$.

(iv) O is centroid

W.K.T $\Delta BOC = \Delta COA = \Delta AOB = \frac{1}{3} \Delta ABC$

Now, $\Delta BOC = \frac{1}{2}(OB)(OC) \sin \angle BOC$



$\Rightarrow \sin \angle BOC = \dfrac{2\,\Delta ABC}{3\ OB \cdot OC}$

|||ly $\sin \angle COA = \dfrac{2\,\Delta ABC}{3\ OC \cdot OA}$

$\sin \angle AOB = \dfrac{2\,\Delta ABC}{3\ OA \cdot OB}$

Lami's theorem is

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB}$$

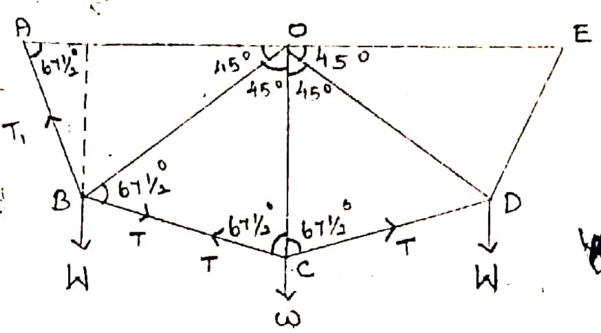
$$\Rightarrow \frac{P \cdot 3OB \cdot OC}{2\Delta ABC} = \frac{Q \cdot 3OC \cdot OA}{2\Delta ABC} = \frac{R \cdot 3OA \cdot OB}{2\Delta ABC}$$

$\div$ by OA OB OC $\quad \dfrac{P}{OA} = \dfrac{Q}{OB} = \dfrac{R}{OC}$

$P : Q : R = OA : OB : OC$

16) Weights W, w, W are attached to points B, C, D (79) respectively of a light string AE where B, C, D divide the string into 4 equal lengths. If the String hangs in the form of 4 Consecutive sides of a regular octagon with the ends A and E attached to points on the same level, S.T W = (√2 ...

Soln:



Let ABCDE be the lower half of the octagon each side of the octagon subtends at the Centre O an angle $\frac{180^\circ}{4} = 45^\circ$.

Now ΔOAB, ΔOBC, ΔOCD, ΔODE are isosceles triangles with vertical angles 45° and the base angles 67½

The forces at C are w and the equal tensions T, T (equal due to symmetry)

The angles opposite to w and T are $\angle BCD$, $\angle BCw$.

$\angle BCD = 67\frac{1}{2}^\circ + 67\frac{1}{2}^\circ$

and $\angle BCw = 180^\circ - 67\frac{1}{2}^\circ$

By Lami's theorem, we have

$$\frac{w}{\sin(67\frac{1}{2}^\circ + 67\frac{1}{2}^\circ)} = \frac{T}{\sin(180^\circ - 67\frac{1}{2}^\circ)}$$

$$\Rightarrow \frac{w}{\sin 135^\circ} = \frac{T}{\sin 67\frac{1}{2}^\circ} \rightarrow (1)$$

The forces at B are W, T, T_1

The angle opposite to W and T are $\angle ABC$, $\angle ABW$

ie) $\angle ABC = 67\frac{1}{2}^\circ + 67\frac{1}{2}^\circ$ and $\angle ABW = 180^\circ - 22\frac{1}{2}^\circ$

$\therefore$ By Lami's theorem $\frac{W}{\sin 135^\circ} = \frac{T}{\sin 22\frac{1}{2}^\circ} \rightarrow (2)$

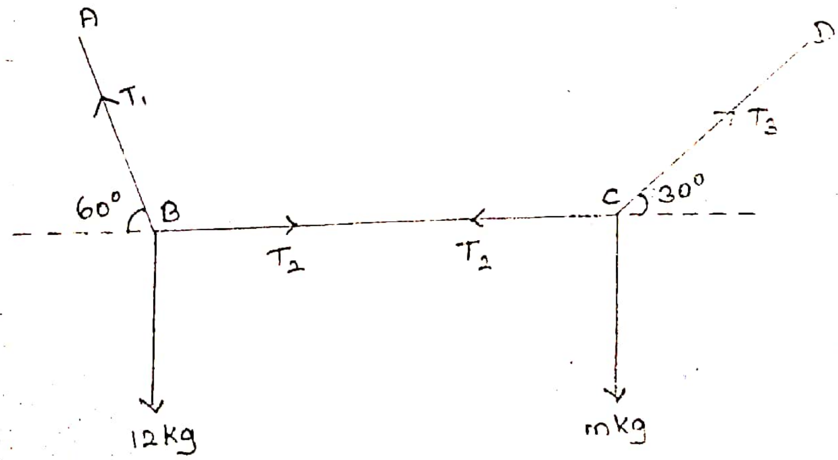
$$\frac{(2)}{(1)} \Rightarrow \frac{W}{w} = \frac{\sin 67\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \frac{\cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} = \frac{1}{\tan 22\frac{1}{2}^\circ}$$

$$= \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{2-1}$$

$$W = (\sqrt{2}+1)w$$

D₁) A string ABCD hangs from fixed points A, D carrying a mass of 12 kg at B and a mass of m kg at C. AB is inclined at 60° to the horizontal, BC is horizontal and CD is inclined at 30° to the horizontal. S.T m=4.

Soln:



Let, The forces acting on the system be T_1, T_2, T_3, 12kg, m kg as shown in figure

By Lami's theorem,

For the forces at B,

$$\frac{T_1}{\sin 90^\circ} = \frac{T_2}{\sin(90^\circ+60^\circ)} = \frac{12\,\text{kg}}{\sin(180^\circ-60^\circ)}$$

$$\Rightarrow \frac{T_2}{\cos 60^\circ} = \frac{12}{\sin 60^\circ}$$

$$T_2 = \frac{12 \times \cos 60^\circ}{\sin 60^\circ}$$

$$= \frac{12}{\frac{2\sqrt{3}}{2}}$$

$$T_2 = \frac{12}{\sqrt{3}} \rightarrow (1)$$

By Lami's theorem,

For the forces at C, we have

$$\frac{T_3}{\sin 90^\circ} = \frac{T_2}{\sin(90^\circ+30^\circ)} = \frac{m}{\sin(180^\circ-30^\circ)}$$

$$\Rightarrow T_2 = \frac{m \cos 30^\circ}{\sin 30^\circ} = \frac{m\sqrt{3}}{2(\frac{1}{2})}$$

$$T_2 = m\sqrt{3} \rightarrow (2)$$

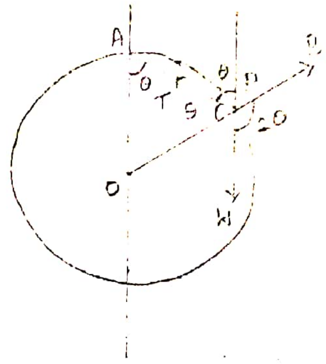
from (1) & (2) $\quad m\sqrt{3} = \frac{12}{\sqrt{3}}$

$$m = \frac{12}{3}$$

$$m = 4\text{ kg}$$

D8) A heavy bead of weight W can slide on a smooth circular wire in a vertical plane. The bead is attached to a light thread to the highest point of the wire, and in equilibrium, the thread is taut and makes an angle θ with the vertical. S.T the tension of the thread and the reaction of the wire on the bead are $2W\cos\theta$ and W. (82)

Soln:



Let A → highest point of the circle

O → Centre of circle

P → bead on the circle

given, $\angle OAP = \theta$

But $OA = OP$ (radius)

$\Rightarrow \triangle OAP$ is isosceles

$\Rightarrow \angle OPA = \theta$

The forces on the bead are

i) Tension T

ii) Normal Reaction R

iii) Weight W

The angles opposite to them are 2θ, $180^\circ-\theta$, $180^\circ-\theta$

By Lami's theorem,

$$\frac{T}{\sin 2\theta} = \frac{R}{\sin(180^\circ-\theta)} = \frac{W}{\sin(180^\circ-\theta)}$$

$$\Rightarrow \frac{T}{\sin 2\theta} = \frac{R}{\sin\theta} = \frac{W}{\sin\theta}$$

$$\Rightarrow \frac{T}{2\cos\theta\sin\theta} = \frac{R}{\sin\theta} = \frac{W}{\sin\theta}$$

$$\Rightarrow \frac{T}{2\cos\theta} = R = W$$

$$T = 2W\cos\theta, \quad R = W$$

D9) A bead of weight W is free to slide on a smooth vertical circle and is connected by a string whose length equals the radius of the circle to the highest point of the circle. Find the tension of the string and the reaction of the circle.

Soln:

In the previous problem, take AP = radius (given)

$\Rightarrow \Delta OAP$ is equilateral triangle

$\Rightarrow \theta = 60^\circ$

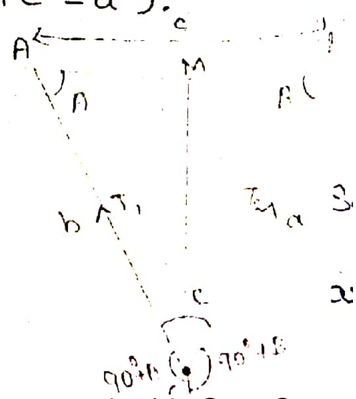
$$\therefore \frac{T}{\sin 120^\circ} = \frac{R}{\sin 120^\circ} = \frac{W}{\sin 120^\circ}$$

$$\therefore T = R = W$$

D10) A and B are two fixed points on a horizontal line at a distance c apart. Two light strings AC, BC of lengths b and a respectively support a mass at C. S.T the tensions of the strings are in the ratio $b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$.

Soln:

Let M be a point vertically below C



Let $\angle BAC = A$, $\angle ABC = B$

The forces acting on the particle are

(i) The weight W vertically downwards

(ii) Tension T_1 along AC

(iii) Tension T_2 along BC

Also the angle between T_1 & T_2 is $\angle C$

ie) $\angle ACB = C$

The tension T_1, T_2 and W keep the particle in equilibrium,

$\therefore$ By Lami's theorem,

$$\frac{T_1}{\sin(90^\circ+B)} = \frac{T_2}{\sin(90^\circ+A)} = \frac{W}{\sin C}$$

Now $\dfrac{T_1}{\cos B} = \dfrac{T_2}{\cos A}$

ie) $$\frac{T_1}{\left(\frac{c^2+a^2-b^2}{2ac}\right)} = \frac{T_2}{\left(\frac{b^2+c^2-a^2}{2bc}\right)}$$

$\div$ by $2abc$ we get

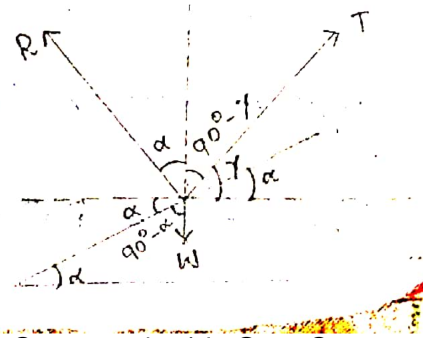
$$\frac{T_1}{b(c^2+a^2-b^2)} = \frac{T_2}{a(b^2+c^2-a^2)}$$

$$T_1 : T_2 = b(c^2+a^2-b^2) : a(b^2+c^2-a^2)$$

D11) A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ. If the slope of the plane be increased to β and the slope of the string be unaltered, the tension of the string is doubled. P.T $\cot\alpha - 2\cot\beta = \tan\gamma$

Soln: Method-I

The forces acting on the particle are



(i) Normal reaction R

(ii) Weight W

(iii) Tension T

Considering the vertical components, we have

$$\therefore \quad R\cos\alpha + T\sin\gamma = W$$

$$R\cos\alpha = W - T\sin\gamma \rightarrow ①$$

From the horizontal components, we get

$$R\sin\alpha = T\cos\gamma \rightarrow ②$$

Dividing ① by ② we get

$$\cot\alpha = \frac{W}{T\cos\gamma} - \tan\gamma \rightarrow ③$$

When α becomes β, T becomes 2T, So

$$\cot\beta = \frac{W}{2T\cos\gamma} - \tan\gamma \rightarrow ④.$$

③ − ④×4 gives

$$\cot\alpha - 2\cot\beta = \tan\gamma$$

Method-II

The forces acting on the particle are

(i) Its weight W vertically downwards

(ii) Normal reaction R

(iii) Tension T along the string

angle between R & T is $90° - (\gamma - \alpha)$

angle between R & W is $180° - \alpha$

angle between W & T is $90° + \gamma$

These forces R, T and W keep the particle in equilibriu

By Lami's theorem, we get

$$\frac{T}{\sin(180^\circ-\alpha)} = \frac{W}{\sin[90^\circ-(\gamma-\alpha)]} = \frac{R}{\sin(90^\circ+\gamma)}$$

$$\frac{T}{\sin\alpha} = \frac{W}{\cos(\gamma-\alpha)}$$

$$T = \frac{W\sin\alpha}{\cos(\gamma-\alpha)} \rightarrow (1)$$

when α is changed to β, T becomes doubled ie) 2T

$$\therefore 2T = \frac{W\sin\beta}{\cos(\gamma-\beta)} \rightarrow (2)$$

(1) & (2)

$$\Rightarrow \frac{2W\sin\alpha}{\cos(\gamma-\alpha)} = \frac{W\sin\beta}{\cos(\gamma-\beta)}$$

$$\Rightarrow 2\sin\alpha\cos(\gamma-\beta) = \sin\beta\cos(\gamma-\alpha)$$

$$\Rightarrow 2\sin\alpha[\cos\gamma\cos\beta+\sin\gamma\sin\beta] = \sin\beta[\cos\gamma\cos\alpha+\sin\gamma\sin\alpha]$$

$$\Rightarrow 2\sin\alpha\cos\gamma\cos\beta + 2\sin\alpha\sin\beta\sin\gamma = \sin\beta\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma$$

$$2\sin\alpha\cos\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma = \cos\alpha\sin\beta\cos\gamma$$

$\div$ by $\sin\alpha\sin\beta\cos\gamma$ we get

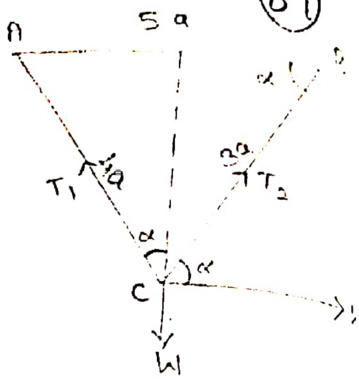
$$2\cot\beta + \tan\gamma = \cot\alpha$$

$$\text{ie)}\quad \cot\alpha - 2\cot\beta = \tan\gamma$$

D12) A particle C of weight W is in equilibrium being supported by two strings CA, CB of length 4a, 3a respectively and acted on by a horizontal force W in the plane ABC. If the ends A, B are at the same level and at a distance 5a apart, S.T the tensions in the strings are $7W/5$, $W/5$.

Soln:

Since $(4a)^2+(3a)^2=(5a)^2$



$\Rightarrow \quad \angle ACB = 90°$

If CA is inclined to the vertical at an angle α, then

$$\cos\alpha = \frac{3}{5}, \quad \sin\alpha = \frac{4}{5} \rightarrow (1)$$

T_1 & T_2 are tensions along CA & CB resolving horizontally to the right and vertically upwards,

$$T_2\cos\alpha - T_1\sin\alpha + W = 0$$

$$T_2\sin\alpha + T_1\cos\alpha - W = 0$$

using (1) $\Rightarrow$ $T_2\left(\frac{3}{5}\right) - T_1\left(\frac{4}{5}\right) + W = 0$

$$T_2\left(\frac{4}{5}\right) + T_1\left(\frac{3}{5}\right) - W = 0$$

$\Rightarrow 3T_2 - 4T_1 + 5W = 0$ and

$\Rightarrow 4T_2 + 3T_1 - 5W = 0$

$$T_1 = \frac{7W}{5}, \quad T_2 = \frac{W}{5}$$

Hence Proved.